Electrostatic forces on a conducting sphere due to a charge on a dielectric half-space

1976 J. Phys. A: Math. Gen. 9 1939

(http://iopscience.iop.org/0305-4470/9/11/018)

J. Phys. A: Math. Gen., Vol. 9, No. 11, 1976. Printed in Great Britain. © 1976

Electrostatic forces on a conducting sphere due to a charge on a dielectric half-space

J S Berry

School of Physical Sciences, The New University of Ulster, Coleraine, Co. Londonderry, N. Ireland, BT52 1SA, UK

Received 5 April 1976, in final form 12 July 1976

Abstract. The electrostatic force on an earthed conducting sphere positioned symmetrically above a circular disc of charge placed on a dielectric half-space is calculated. Bispherical polar coordinates are used when the sphere does not touch the half-space and degenerate bipolar coordinates are used when the sphere touches the half-space. Results are given in terms of rapidly converging infinite series in the former and infinite integrals in the latter. The dependence of the force on sphere radius and sphere–half-space separation is presented graphically.

1. Introduction

In a previous paper, Berry and Higginbotham (1975 to be referred to as BH) employed the method of images to evaluate the electrostatic force on a conducting sphere due to a circular disc of charge on an infinitely thin insulating plane. The main restriction of the BH theory is that it applies only to an insulating material of infinitesimal thickness.

This present paper investigates the electrostatic force on a conducting sphere due to a disc of charge placed on the surface of a dielectric half-space. Two cases are considered which require different coordinate systems. For the case of the sphere not touching the half-space, bispherical polar coordinates are used; Laplace's equation is solved in the two regions (i) exterior to the sphere and half-space and (ii) within the half-space, and the boundary conditions are used to match both solutions at the surface of the half-space. When the sphere is touching the half-space, degenerate bipolar coordinates are used. In each geometry the solutions of Laplace's equation are readily available, see for instance Lebeder *et al* (1965). In the former the solution is given in terms of Legendre polynomials whereas in the latter Bessel functions are used.

The electrostatic force on the sphere is given by an integral expression which is evaluated numerically. If the half-space has unit dielectric constant, the problem is identical with that solved in the previous paper, and the solution obtained here is the same thus confirming the previous results.

2. Bispherical polar coordinates

The system of bispherical polar coordinates (η, τ, ϕ) is based on the properties of sets of coaxial circles as illustrated in figure 1.

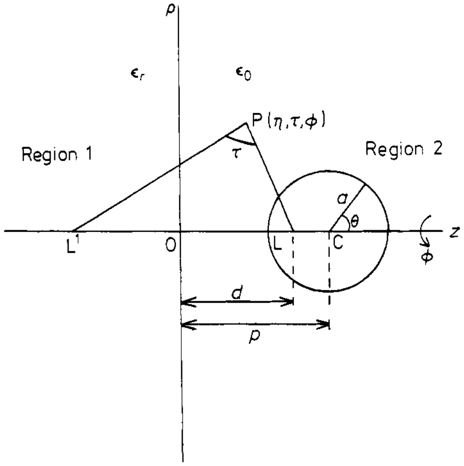

Figure 1. The system of bispherical polar coordinates (η, τ, ϕ) employed in the analysis. The half-space (region 1) carries a uniform surface charge density σ within a disc of radius c. The earthed conducting sphere is represented by the circle, centre C.

The points L and L^1 are at a distance d from the origin O; the variable η is defined by $\eta = -\ln(PL/PL^1)$ and τ is the angle L^1PL. The locus of P such that η remains constant, with τ running between $-\pi$ and π, is a circle centre C. The coordinates (ρ, ϕ, z) form the cylindrical polar coordinate system which are related to (η, τ, ϕ) by

$$z = \frac{d \sinh \eta}{(\cosh \eta - \cos \tau)}, \qquad \rho = \frac{d \sin \tau}{(\cosh \eta - \cos \tau)}$$

and by geometry $d = r \sinh \eta$, $p = r \cosh \eta$; where r is the radius of the circle corresponding to constant η. The plane $z = 0$ is given by $\eta = 0$.

Solutions of Laplace's equation are separable with weight function $(\cosh \eta - \cos \tau)^{1/2}$ and, if the problem is symmetric in ϕ, are given by

$$(\cosh \eta - \cos \tau)^{1/2} \sum_{n=0}^{\infty} (A_n \cosh(n+\tfrac{1}{2})\eta + B_n \sinh(n+\tfrac{1}{2})\eta) P_n(\cos \tau)$$

where $P_n(\eta)$ are the Legendre polynomials.

Consider an earthed sphere of radius a whose centre is distance $p(>a)$ from a dielectric half-space of dielectric constant ϵ_r. In this coordinate system the sphere is denoted by $\{\eta = \eta_0(>0), -\pi < \tau < \pi, 0 < \phi < 2\pi\}$ and the half-space (region 1) is given by $\{-\infty < \eta < 0, -\pi < \tau < \pi, 0 < \phi < 2\pi\}$. The region 2 outside the sphere and half-space is vacuum ($\epsilon_r = 1$) and is given by $\{0 < \eta < \eta_0, -\pi < \tau < \pi, 0 < \phi < 2\pi\}$. A circular area, centre O, radius c and carrying a charge density σ is placed on the surface of the half-space.

The electrostatic potentials in regions 1 and 2 are respectively given by

$$V_1 = (\cosh \eta - \cos \tau)^{1/2} \sum_{n=0}^{\infty} A_n \, e^{(n+\frac{1}{2})\eta} P_n(\cos \tau),$$

$$V_2 = (\cosh \eta - \cos \tau)^{1/2} \sum_{n=0}^{\infty} B_n \sinh[(n+\tfrac{1}{2})(\eta - \eta_0)] P_n(\cos \tau)$$

where $\cosh \eta_0 = p/a$ and $d = a \sinh \eta_0$.

The choice of V_2 is such that on the sphere, $V_2=0$ as required, so that one boundary condition is immediately satisfied. On the interface between regions 1 and 2 we require

$$V_1=V_2, \qquad \eta=0$$

$$\epsilon_r\frac{\partial V_1}{\partial z}-\frac{\partial V_2}{\partial z}=\begin{cases}\sigma/\epsilon_0 & 0<\rho<c \\ 0 & \rho>c\end{cases} \qquad \eta=0.$$

The second of these equations has been given in terms of cylindrical polar coordinates; however, it is easy to show that on $\eta=0$, $\partial V/\partial z=d^{-1}(1-\cos\tau)(\partial V/\partial\eta)$ and the range $0<\rho<c$ becomes $\pi>\tau>\tau_0$ where $\cos\tau_0=(c^2-p^2+a^2)/(c^2+p^2-a^2)$.

On substitution for V_1 and V_2 the constants A_n and B_n become

$$\epsilon_0B_n=\frac{d\sigma}{\cosh[(n+\frac{1}{2})\eta_0]+\epsilon_r\sinh[(n+\frac{1}{2})\eta_0]}\int_{-1}^{\cos\tau_0}\frac{P_n(\mu)}{(1-\mu)^{3/2}}\,\mathrm{d}\mu$$

and $A_n=-B_n\sinh[(n+\frac{1}{2})\eta_0]$.

3. Attractive force on the sphere

The force on the sphere towards the plane is given by

$$F=\iint_{\text{sphere}}\frac{\sigma_s^2}{2\epsilon_0}\cos\theta\,\mathrm{d}S$$

where σ_s is the charge density on the surface of the sphere $(=\epsilon_0\partial V_2/\partial r)$, $\mathrm{d}S=a^2\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\psi$ and (r,θ,ψ) are spherical polar coordinates centred on C. In terms of bispherical polar coordinates it can be shown that

$$\left.\frac{\partial V_2}{\partial r}\right|_{r=a}=\frac{(\cosh\eta_0-\cos\tau)}{a\sinh\eta_0}\left.\frac{\partial V_2}{\partial\eta}\right|_{\eta=\eta_0},$$

so that

$$\sigma_s=\frac{(\cosh\eta_0-\cos\tau)^{3/2}}{a\sinh\eta_0}\sum_{n=0}^{\infty}(n+\tfrac{1}{2})B_nP_n(\cos\tau);$$

and, when $r=a$, $\cos\theta=(\cosh\eta_0\cos\tau-1)/(\cosh\eta_0-\cos\tau)$.

The force on the sphere thus becomes

$$F=\pi\sigma^2I/\epsilon_0$$

where

$$I=\int_{-1}^{+1}(\mu\cosh\eta_0-1)\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}(k+\tfrac{1}{2})(n-k+\tfrac{1}{2})\epsilon_0B_k\epsilon_0B_{n-k}P_k(\mu)P_{n-k}(\mu)\,\mathrm{d}\mu\right).$$

Using the orthogonality conditions for the Legendre polynomials, this reduces to

$$I=\cosh\eta_0\sum_{m=0}^{\infty}(m+1)\epsilon_0B_m\epsilon_0B_{m+1}-\sum_{m=0}^{\infty}\tfrac{1}{2}(2m+1)(\epsilon_0B_m)^2.$$

The convergence of these series is sufficiently rapid for a reliable value to be computed from the first twenty terms. Given values of a, p, c and σ, the attractive force on the sphere can be calculated.

4. Degenerate bipolar coordinates

As the sphere approaches the plane, $p/a \to 1$, $\eta_0 \to \infty$ and the analysis given above, although still correct, leads to much slower convergence of the series I. It is much more convenient to use degenerate bipolar coordinates to solve the problem of an earthed sphere in contact with a dielectric half-space on which there is a surface charge density. These are constructed from the properties of sets of coaxial circles which touch the plane $z = 0$.

In terms of cylindrical polar coordinates (ρ, z, ϕ) the degenerate bipolar coordinates are given by

$$z = \frac{\beta}{\alpha^2 + \beta^2}, \qquad -\infty < \beta < \infty, 0 \leqslant \alpha < \infty$$

$$\rho = \frac{\alpha}{\alpha^2 + \beta^2}.$$

The locus of a point such that $\beta = \beta_0 =$ constant represents a circle centre C and radius $\frac{1}{2}\beta_0$. The plane $z = 0$ is given by $\beta = 0$. The dielectric half-space (region 1) is denoted by $\{\beta < 0, 0 \leqslant \alpha < \infty, 0 \leqslant \phi < 2\pi\}$ and the region 2 outside the half-space and sphere is given by $\{0 < \beta < \beta_0, 0 \leqslant \alpha < \infty, 0 \leqslant \phi < 2\pi\}$ (see figure 2).

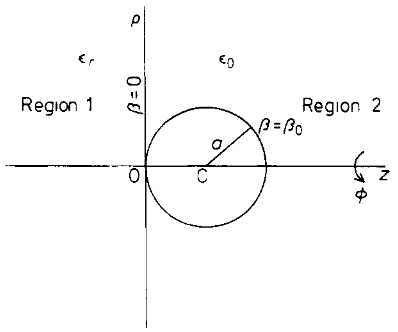

Figure 2. The geometry of the sphere touching the half-space. The degenerate bipolar coordinates (α, β, ϕ) are related to the cylindrical polar coordinates shown in the figure by $z = \beta/(\alpha^2 + \beta^2)$, $\rho = \alpha/(\alpha^2 + \beta^2)$.

Solutions of Laplace's equation are separable with weight function $(\alpha^2 + \beta^2)^{1/2}$ and, if the problem is symmetric in ϕ, are given by

$$(\alpha^2 + \beta^2)^{1/2} \int_{\lambda=0}^{\infty} (A_\lambda \cosh \lambda\beta + \beta_\lambda \sinh \lambda\beta) J_0(\lambda\alpha)\, d\lambda$$

where $J_0(\lambda\alpha)$ is the Bessel function of the first kind of zero order.

Repeating the analysis of § 2, the electrostatic potentials in regions 1 and 2 are respectively

$$V_1 = (\alpha^2 + \beta^2)^{1/2} \int_{\lambda=0}^{\infty} J_0(\lambda\alpha) A_\lambda\, e^{\lambda\beta}\, d\lambda \qquad -\infty < \beta \leqslant 0$$

and

$$V_2=(\alpha^2+\beta^2)^{1/2}\int_{\lambda=0}^{\infty} J_0(\lambda\alpha)B_\lambda \sinh\lambda(\beta-\beta_0)\,\mathrm{d}\lambda \qquad 0\leqslant\beta\leqslant\beta_0$$

where

$$\epsilon_0 B_\lambda=\frac{\sigma}{(\cosh(\lambda/2a)+\epsilon_r\sinh(\lambda/2a))}\int_{1/c}^{\infty}\frac{J_0(\lambda\alpha)}{\alpha^2}\,\mathrm{d}\alpha$$

and $A_\lambda=-B_\lambda\sinh(\lambda/2a)$.

The integral $c_\lambda=\int_{1/c}^{\infty}[J_0(\lambda\alpha)/\alpha^2]\,\mathrm{d}\alpha$ can be evaluated using the recurrence relations for Bessel functions and known integrals (see for instance Gradshteyn and Ryzhik 1965). Thus

$$c_\lambda=cJ_0(\lambda/c)-\lambda J_1(\lambda/c)-\lambda+2\lambda\sum_{k=0}^{\infty}J_{2K+1}(\lambda/c).$$

5. Force on an earthed conducting sphere touching the dielectric half-space

The force on the sphere towards the half-space is given in terms of spherical polar coordinates by

$$\iint_{\text{sphere}}(\sigma_s^2/2\epsilon_0)\cos\theta\,\mathrm{d}S.$$

In this case

$$\sigma_s=\left(\alpha^2+\frac{1}{4a^2}\right)^{3/2}\int_{\lambda=0}^{\infty}\epsilon_0 B_\lambda J_0(\alpha\lambda)\lambda\,\mathrm{d}\lambda$$

and

$$\cos\theta\,\mathrm{d}S=a^2\sin\theta\cos\theta\,\mathrm{d}\theta\,\mathrm{d}\psi=\alpha[(1/4a^2)-\alpha^2]\,\mathrm{d}\alpha\,\mathrm{d}\psi/[\alpha^2+(1/4a^2)]^3.$$

Thus

$$F=\frac{\pi\sigma^2}{\epsilon_0}\int_{\alpha=0}^{\infty}\alpha\left(\frac{1}{4a^2}-\alpha^2\right)\left(\int_{\lambda=0}^{\infty}\frac{c_\lambda J_0(\alpha\lambda)\lambda\,\mathrm{d}\lambda}{[\cosh(\lambda/2a)+\epsilon_r\sinh(\lambda/2a)]}\right)^2\mathrm{d}\alpha.$$

This expression is evaluated numerically on the computer and each integral converges sufficiently rapidly so that the upper limits in each case may be taken to be around twenty.

6. Results of calculations

Figure 3 shows the dependence of the force as a function of the sphere–half-space distance $(p-a)$. In like manner to BH, F/K is plotted as ordinate where $K=\pi\sigma^2/4\epsilon_0$; the value of $c=1$, and σ is the total charge density on the surface of the dielectric. If a charge density σ^1 is placed on the surface of the dielectric then, of course, σ is the effective total charge density σ^1/ϵ_r. A value of $a=0\cdot5$ was chosen to illustrate the results, it should be noted that each curve will intersect the ordinate at zero slope for a

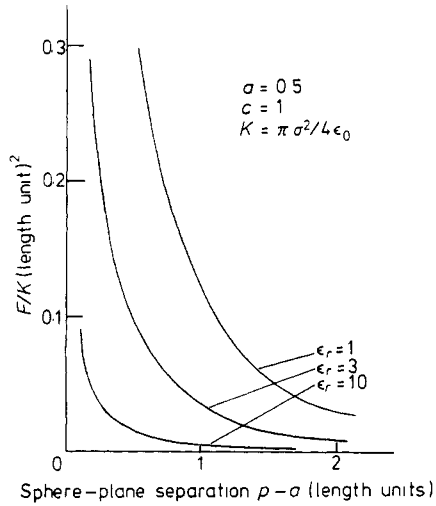

Figure 3. The forces on an earthed conducting sphere as functions of sphere–plane separation $(p - a)$.

finite value of F/K, however the scaling makes this difficult to illustrate. The values of F/K when $(p - a)$ is zero can be obtained from figure 4 and, if required, a continuation of the curves in figure 3 for small spacings can easily be drawn.

Figure 4 shows the dependence of the force on the sphere radius when the sphere touches the half-space. Again $c = 1$. It should be noted that each of the curves tends to 2 as $a \to \infty$.

The curves have been drawn for one value of $c (=1)$. However if F is the force on a sphere of radius a_1 due to a disc of charge of radius c_1 and centre a distance p_1 from the half-space, then $F = c_1^2 F_1$; where F_1 is the force on a sphere of radius a_1/c_1 due to a disc of charge of radius 1 and centre a distance p_1/c_1 from the half-space.

Finally, figure 5 shows the dependence of the force against c, on a sphere of radius 0·5 when its centre is a distance 1·0 from the half-space and for $\epsilon_r = 1{\cdot}0$. The maximum

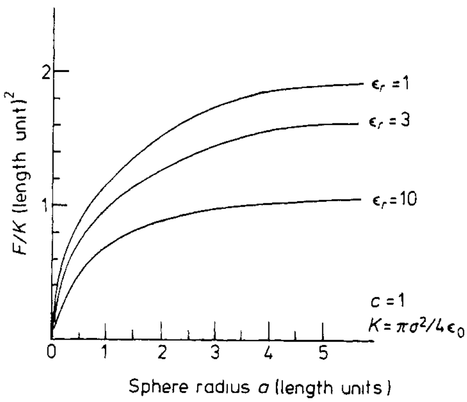

Figure 4. The forces on an earthed conducting sphere as functions of the sphere radii a for the case where the sphere touches the half-space.

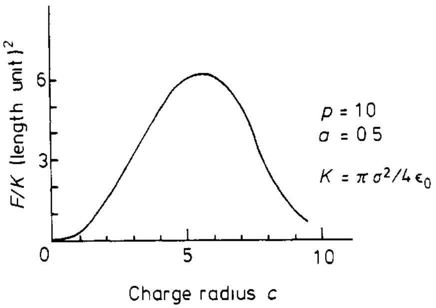

Figure 5. The forces on an earthed conducting sphere as a function of c, when $a = 0{\cdot}5$ and $p = 1{\cdot}0$.

value of the force occurs for $c = 5{\cdot}7$; this maximum value of c will obviously depend on the parameters a and p. It should be noted that as $c \to \infty$, $F \to 0$ which is in accordance with physical arguments.

7. Conclusion

This paper extends the results for the electrostatic force on an earthed conducting sphere due to a charge on a thin insulating plane, to the force due to a charge on a dielectric half-space of dielectric constants 3·0 and 10·0. Two coordinate systems have been used and the forces are given in terms of rapidly converging integrals in one case and rapidly converging infinite series in the other. The solutions are given graphically.

Acknowledgments

I am grateful to Dr I G Higginbotham for suggesting the problem and for his helpful suggestions in the preparation of this paper. I would also like to thank one of the referees for finding an error in the original work and for his suggestions in improving the paper.

References

Berry J S and Higginbotham I G 1975 *J. Phys. A: Math. Gen.* **8** 1842–51

Gradshteyn I S and Ryzhik I M 1965 *Tables of Integrals, Series and Products* (New York and London: Academic)

Lebeder N N, Skalskaya I P and Uflyand Y S 1965 *Problems of Mathematical Physics* (Englewood Cliffs, N.J.: Prentice-Hall)